Pricing of Exotic Options via Conditional Expectations

Math 622

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1 Knock out Barrier option

Reading material: Ocone's Lecture note 5 part 1, Shreve's 7.3.3
Let $S(t)$ satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider the Knock-out Barrier option with barrier b and strike price K:

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \leq B\}}.$$

The risk-neutral price $V(t)$ can be expressed as

$$\begin{aligned} V(t) &= \mathbb{E}\Big[e^{-r(T-t)} V_T | \mathcal{F}(t)\Big] \\ &= \mathbf{1}_{\{\max_{[0,t]} S_t \leq B\}} \mathbb{E}\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_t \leq B\}} | S(t)\Big]. \end{aligned}$$

To obtain an explicit formula for $V(t)$, we need to evaluate

$$\begin{aligned} &\mathbb{E}\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_t \leq B\}} | S(t)\Big] \\ &= \mathbb{E}\Big[e^{-r(T-t)}(S_T - K)^+ | S(t)\Big] - \mathbb{E}\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_t > B\}} | S(t)\Big]. \quad (1) \end{aligned}$$

Since $\mathbb{E}\Big[e^{-r(T-t)}(S_T - K)^+ | S(t)\Big]$ is already given by Black-Scholes formula, *we only need to evaluate*

$$\begin{aligned} w(t,x) &:= \mathbb{E}\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_t > B\}} | S(t) = x\Big] \\ &:= \mathbb{E}\Big[e^{-r(T-t)}(S_T - K) \mathbf{1}_{S_T \geq K} \mathbf{1}_{\{\max_{[t,T]} S_t > B\}} | S(t) = x\Big] \quad (2) \end{aligned}$$

Remark 1.1. *The split in Equation* (1) *may seem non-intuitive. We will see why later. It suffices for now to observe that to compute the Expectation we need the joint distribution between 2 Random variables:* $S(T)$ *and* $Y(t) := \max_{[t,T]} S(t)$. *Because of their interaction, it is easier to compute their joint distribution in the form* $\mathbb{P}(S(T) \geq x, Y(t) \geq y)$ *than* $\mathbb{P}(S(T) \geq x, Y(t) \leq y)$.

1.1 Step 1: A first rewrite of $w(t,x)$

Denote

$$\alpha := \frac{r - \frac{1}{2}\sigma^2}{\sigma}.$$

Then for $s \geq t$

$$S(s) = S(t) \exp\Big[\sigma\big(W(s) - W(t)\big) + \alpha(s-t)\big)\Big].$$

The term inside the exponential (modulo the σ) is just a Brownian motion with drift starting at time t. So we denote it by a new name to reflect this fact:

$$\begin{aligned} \widehat{W}(u) &:= W(t+u) - W(t) + \alpha u, u \geq 0 \\ \widehat{M}(u) &:= \max_{s\in[0,u]} \widehat{W}(s). \end{aligned}$$

Note that for $s \geq t$

$$\max_{u\in[t,s]} S(u) = S(t)e^{\sigma\widehat{M}(s-t)}.$$

Then for $s \geq t$ we have

$$\begin{aligned} S(s) &= S(t)e^{\sigma\widehat{W}(s-t)} \\ \mathbf{1}_{S(s)\geq K} &= \mathbf{1}_{\widehat{W}(s-t)\geq\frac{\log(K/S(t))}{\sigma}} \\ \mathbf{1}_{\max_{u\in[t,s]} S(u)>B} &= \mathbf{1}_{\widehat{M}(s-t)>\frac{\log(B/S(t))}{\sigma}} \end{aligned}$$

Then substituing this into Equation (2), replacing $S(t) = x$ gives

$$\begin{aligned} w(t,x) &= e^{-r(T-t)}\mathbb{E}\Big[\big(xe^{\sigma\widehat{W}(T-t)} - K\big)\mathbf{1}_{\{\widehat{W}(T-t)\geq\frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)>\frac{\log(B/x)}{\sigma}\}}\Big] \\ &= xe^{-r(T-t)}\mathbb{E}\Big[e^{\sigma\widehat{W}(T-t)}\mathbf{1}_{\{\widehat{W}(T-t)\geq\frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)>\frac{\log(B/x)}{\sigma}\}}\Big] \\ &\quad -Ke^{-r(T-t)}\mathbb{E}\Big[\mathbf{1}_{\{\widehat{W}(T-t)\geq\frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)>\frac{\log(B/x)}{\sigma}\}}\Big]. \end{aligned} \tag{3}$$

Remark 1.2. *Note that the expression in* (3) *involves the distribution of a Brownian motion with drift and its running maximum. Studying Brownian motion with drift is inconvenient. But by applying a change of measure (via Girsanov's theorem), we can find a different measure such that under it,* $\widehat{W}$ *is a BM. So that's our next step.*

1.2 Apply Girsanov's Theorem to transform $\widehat{W}$ into a BM

Since the expression (3) involves only expectation, we only need to pay attention to the distribution of $\widehat{W}(t)$. So we can assume there exists a Brownian motion $\widetilde{W}(u), 0 \leq u \leq T-t$ (tilde to distinguish with our original BM W) such that

$$\widehat{W}(u) = \widetilde{W}(u) + \alpha u, u \in [0, T-t].$$

Since $\widehat{W}$ has drift term αt, our change of measure kernel is

$$Z(T-t) = \exp[-\alpha \widetilde{W}(T-t) - \frac{\alpha^2}{2}(T-t)].$$

Denoting our original measure as $\mathbb{P}$ and define

$$d\mathbb{Q} := Z(T-t)d\mathbb{P},$$

then note that

$$\begin{aligned} d\mathbb{P} &= Z(T-t)^{-1}d\mathbb{Q} \\ &= \exp[\alpha \widehat{W}(T-t) - \frac{\alpha^2}{2}(T-t)], \end{aligned}$$

So that

$$\begin{aligned} w(t,x) &= xe^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma \widehat{W}(T-t)}\mathbf{1}_{\{\widehat{W}(T-t)\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)> \frac{\log(B/x)}{\sigma}\}}e^{\alpha\widehat{W}(T-t)-\frac{\alpha^2}{2}(T-t)}\right] \\ &\quad -Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\widehat{W}(T-t)\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)> \frac{\log(B/x)}{\sigma}\}}e^{\alpha\widehat{W}(T-t)-\frac{\alpha^2}{2}(T-t)}\right] \\ &= xe^{-(r+\frac{\alpha^2}{2})(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{(\alpha+\sigma)\widehat{W}(T-t)}\mathbf{1}_{\{\widehat{W}(T-t)\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)> \frac{\log(B/x)}{\sigma}\}}\right] \\ &\quad -Ke^{-(r+\frac{\alpha^2}{2})(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha\widehat{W}(T-t)}\mathbf{1}_{\{\widehat{W}(T-t)\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}(T-t)> \frac{\log(B/x)}{\sigma}\}}\right], \end{aligned} \tag{4}$$

where now *what we have gained is $\widehat{W}$ is a Brownian motion under $\mathbb{Q}$.*

Remark 1.3. *Note that the expression* (4) *is just some Expectation involving the exponential of BM, over the set where the BM and its running maximum are larger than some values. When we study barrier option with different paramters, such Expectation will appear very often (and possibly in other contexts in which the running max is involved, say the Lookback Option). Think of the function $c(t,x)$ in the Black-Scholes formula and how it appears in our evaluation of the Euro call option in Chapter 11 for example. So it is convenient for us to evaluate that Expectation in the most general form, with as many free paramaters as possible. Then in the special cases we encounter in pricing options, we only need to plug those parameters into our known function to obtain the answer. This is the next and last step in pricing the Knockout Barrier option by conditional Expectation.*

1.3 The function $H_s(\alpha, \beta, k, b)$

Let $W(t)$ be a Brownian motion and $M(t) := \max_{[0,t]} W(s)$ its running maximum. Define

$$H_s(\alpha, \beta, k, b) := E\Big[\mathbf{1}_{\{W(s)\geq k\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)+\beta M(s)}\Big].$$

$H_s(\alpha, \beta, k, b)$ has an explicit form for certain range of parameters (α, β, k, b) that we will discuss in part 2 of Lecture 5 notes. For now let's just observe the following:
(i) Since $M(s) \geq 0$ by definition, we only need to consider $b \geq 0$.
(ii) Since $M(s)$ and $W(s)$ are continuous random variables (a fact we will establish for $M(s)$) we have

$$\begin{aligned} H_s(\alpha, \beta, k, b) &:= E\Big[\mathbf{1}_{\{W(s)\geq k\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)+\beta M(s)}\Big] \\ &:= E\Big[\mathbf{1}_{\{W(s)\geq k\}}\mathbf{1}_{\{M(s)\geq b\}}e^{\alpha W(s)+\beta M(s)}\Big] \\ &:= E\Big[\mathbf{1}_{\{W(s)> k\}}\mathbf{1}_{\{M(s)>b\}}e^{\alpha W(s)+\beta M(s)}\Big] \\ &:= E\Big[\mathbf{1}_{\{W(s)> k\}}\mathbf{1}_{\{M(s)\geq b\}}e^{\alpha W(s)+\beta M(s)}\Big]. \end{aligned}$$

Then we have

$$\begin{aligned} w(t,x) &= e^{-(r+\frac{\alpha^2}{2})(T-t)}\Big[xH_{T-t}\Big(\alpha+\sigma, 0, \frac{\log(K/x)}{\sigma}, \frac{\log(B/x)}{\sigma}\Big) \\ &\quad -KH_{T-t}\Big(\alpha, 0, \frac{\log(K/x)}{\sigma}, \frac{\log(B/x)}{\sigma}\Big)\Big]. \end{aligned}$$

and the original Knockout Barrier option price is:

$$V(t) = \mathbf{1}_{\{\max_{[0,t]} S_t \leq B\}}\big[c(t, S(t)) - w(t, S(t))\big],$$

where

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

is given by Black-Scholes formula.

2 Lookback Option

Reading material: Ocone's Lecture note 5 part 2, Shreve's 7.4

2.1 Preliminary discussion

Let $S(t)$ satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider the Knock-out Barrier option with barrier b and strike price K:

$$V_T = \max_{[0,T]} S_t - S_T.$$

The risk-neutral price $V(t)$ can be expressed as

$$\begin{aligned} V(t) &= \mathbb{E}\Big[e^{-r(T-t)} V_T | \mathcal{F}(t)\Big] \\ &= \mathbb{E}\Big[e^{-r(T-t)} \max_{[0,T]}\{S_t\} - S_T | \mathcal{F}(t)\Big] \\ &= \mathbb{E}\Big[e^{-r(T-t)} \max_{[0,T]}\{S_t\} | \mathcal{F}(t)\Big] - S(t). \end{aligned}$$

Now to do further analysis (to reduce Conditional Expectation to an Expectation via the Independence Lemma) we may want to separate the term $\max_{[0,T]}\{S_t\}$ into some expression involving $S_u, u \in [0,t]$ and $S_u, u \in [t,T]$. One way to do this is

$$\max_{[0,T]}\{S_t\} = \max_{[0,t]}\{S_t\} \vee \max_{[t,T]}\{S_t\}.$$

So far so good, but we need to do more work here, since the operator $\vee$ does not "factor out" of the conditional expectation (we cannot factor $\max_{[0,t]}\{S_t\}$ out of $E(.|\mathcal{F}(t))$). Looking at this in another way, the running max of $S(t)$:
$Y(t) = \max_{[0,t]} S(u)$ is not a Markov process.
However, there is a usual approach in studying Markov process like this: If $X(t)$ is not a Markov process, by increasing the components of $X(t)$, we may still yet obtain a Markov process.
In this case, we consider the two-component process $(S(t), Y(t))$ instead of just $Y(t)$. Then for $s > t$

$$Y(s) = \max\{Y(t), \max_{[t,s]} S(u)\} = \max\{Y(t), S(t)e^{\sigma \widehat{M}(s-t)}\},$$

where recall that we defined in Section 1

$$\begin{aligned} \alpha &:= \frac{r - \frac{1}{2}\sigma^2}{\sigma} \\ \widehat{W}(u) &:= W(t+u) - W(t) + \alpha u, u \geq 0 \\ \widehat{M}(u) &:= \max_{s \in [0,u]} \widehat{W}(s). \end{aligned}$$

Then since $\widehat{M}$ and $\widehat{W}$ are independent of $\mathcal{F}(t)$ under the risk neutral measure, we get that $(S(t), Y(t))$ is a Markov process under this measure as well (how to reach this conclusion is left as a homework exercise).
We then have

$$\begin{aligned} V(t) &= \mathbb{E}\Big[e^{-r(T-t)}V_T|\mathcal{F}(t)\Big] \\ &= \mathbb{E}\Big[e^{-r(T-t)}\max_{[0,T]}\{S_t\}|\mathcal{F}(t)\Big] - S(t) \\ &= \mathbb{E}\Big[e^{-r(T-t)}\max\{Y(t), S(t)e^{\sigma\widehat{M}(T-t)}\}|\mathcal{F}(t)\Big] - S(t) \\ &= v(t, S(t), Y(t)), \end{aligned}$$

where

$$v(t,x,y) = e^{-r(T-t)}\mathbb{E}\Big[\max(y, xe^{\sigma\widehat{M}(T-t)})\Big] - x.$$

2.2 Apply Girsanov

Similar to the discussion in section 1, $v(t,x,y)$ involves the distribution of the running max of a BM with drift, so we want to apply Girsanov's theorem to transform it to a BM. The result is

$$v(t,x,y) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[e^{\alpha\widehat{W}(T-t)-\frac{\alpha^2}{2}(T-t)}\max(y, xe^{\sigma\widehat{M}(T-t)})\Big] - x,$$

where $\widehat{W}$ now is a BM under $\mathbb{Q}$.
Note that the expression inside expectation is not (yet) of the form provided by the function $H_s(\alpha,\beta,k,b)$. Noting the fact that $x > 0$ since it is the stock price $S(t)$, we have

$$\begin{aligned} \max(y, xe^{\sigma\widehat{M}(T-t)}) = y \quad &\text{if} \quad \widehat{M}(T-t) < \frac{1}{\sigma}\log(y/x) \\ \max(y, xe^{\sigma\widehat{M}(T-t)}) = xe^{\sigma\widehat{M}(T-t)} \quad &\text{if} \quad \widehat{M}(T-t) \geq \frac{1}{\sigma}\log(y/x). \end{aligned}$$

Denoting

$$b := \frac{1}{\sigma}\log(y/x),$$

and note that the domain of interest for $v(t,x,y)$ is $y \geq x > 0$ thus $b \geq 0$. Then

$$\begin{aligned} \max(y, xe^{\sigma\widehat{M}(T-t)}) &= y\mathbf{1}_{\widehat{M}(T-t)<b} + xe^{\sigma\widehat{M}(T-t)}\mathbf{1}_{\widehat{M}(T-t)\geq b} \\ &= y + \Big[xe^{\sigma\widehat{M}(T-t)} - y\Big]\mathbf{1}_{\widehat{M}(T-t)\geq b}. \end{aligned}$$

Plug this back into the expectation, coupled with the fact that

$$\mathbb{E}^{\mathbb{Q}}\left[ye^{\alpha\widehat{W}(T-t)-\frac{\alpha^2}{2}(T-t)}\right] = y,$$

after simplification we have

$$\begin{aligned} v(t,x,y) &= e^{-r(T-t)}y - x + xe^{-(r+\frac{\alpha^2}{2})(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\widehat{M}(T-t)\geq b\}}e^{\alpha\widehat{W}(T-t)+\sigma\widehat{M}(T-t)}\right] \\ &\quad -ye^{-(r+\frac{\alpha^2}{2})(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\widehat{M}(T-t)\geq b\}}e^{\alpha\widehat{W}(T-t)}\right] \\ &= e^{-r(T-t)}y - x + xe^{-(r+\frac{\alpha^2}{2})(T-t)}H_{T-t}\big(\alpha,\sigma,-\infty,b\big) \\ &\quad -ye^{-(r+\frac{\alpha^2}{2})(T-t)}H_{T-t}\big(\alpha,0,-\infty,b\big). \end{aligned} \tag{5}$$

Remark 2.1. *You may question why we go into such great length to derive the long expression for $v(t,x,y)$ in this section or $v(t,x)$ in section 1. You may argue that once we have the expression of $v(t,x,y)$ as an expectation, we can just simulate the paths of $S(t)$ and take the average to obtain the expectation. This is correct. However, the work that we have done in expressing $v(t,x,y)$ in the form of (5) can be very helpful in increasing the efficiency of the computation. We have "simplified" the computation (not in the expression, of course, but in the actual computation time). The reason is the function $H_s(\alpha,\beta,k,b)$ is found explicitly via the cumulative distribution of the standard normal, which we have very efficient algorithms to compute. On the other hand, if you simulate the paths of $S(t)$, also taking into account its running max or when it's knocked out, you will see that unless you use some good algorithm, the efficiency of just taking the average won't be too great.*